A FORMULA WITH SOME APPLICATIONS
TO THE THEORY OF LYAPUNOV EXPONENTS*

BY

ARTUR AVILA

Collège de France, 3 rue d'Ulm, 75005 Paris, France
e-mail: avila@impa.br

AND

JAIRO BOCHI

IMPA, Estr. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil
e-mail: bochi@impa.br

ABSTRACT

We prove an elementary formula about the average expansion of certain products of 2 by 2 matrices. This permits us to quickly re-obtain an inequality by M. Herman and a theorem by Dedieu and Shub, both concerning Lyapunov exponents. Indeed, we show that equality holds in Herman's result. Finally, we give a result about the growth of the spectral radius of products.

1. Introduction

A major problem in smooth ergodic theory is to determine whether a given measure-preserving diffeomorphism has a non-zero Lyapunov exponent. This problem is also of interest in the more general setting of linear cocycles. However, it is difficult to show the existence of non-zero exponents without strong conditions like uniform hyperbolicity.

In [He], Herman devised a method to bound the upper Lyapunov exponent of some cocycles from below and constructed the first examples of non-uniformly hyperbolic two-dimensional systems with a positive exponent. Such examples are very delicate: it is shown in [Bo] that the exponent of non-hyperbolic cocycles drops to zero with an arbitrarily small C^0-perturbation of the cocycle.

One of the methods of Herman estimates the average of the upper Lyapunov exponent of systems in a special parametrized family. While each individual system may be unstable, this average estimate is robust. Using Herman's estimate, Knill proved in [Kn] that among bounded measurable $\mathrm{SL}(2,\mathbb{R})$-cocycles those with a positive exponent are dense.

This idea — to consider systems included in some suitable family and to show that global properties of the family imply good properties for many individual elements — is also present in the recent paper [DS]. This reasoning has been conjectured to work in more generality in [BPSW].

We will consider the following situation: take matrices $A_1,\ldots,A_n$ in $\mathrm{SL}(2,\mathbb{R})$. Of course, the norm $\|A_n \cdots A_1\|$ can be much smaller than $\prod \|A_j\|$. Now we put those matrices inside a family parametrized by a circle: $A_{j,\theta} = A_j R_\theta$ (we indicate by R_θ a rotation of angle θ). Instead of looking at the norms, we will deal with the related quantity

$$N(A) = \log\left(\frac{\|A\| + \|A\|^{-1}}{2}\right).$$

In this note we will prove:

$$\frac{1}{2\pi}\int_0^{2\pi} N(A_{n,\theta}\cdots A_{1,\theta})d\theta = \sum_{j=1}^n N(A_j).$$

In particular, if the $\|A_j\|$ are large then $\|A_{n,\theta}\cdots A_{1,\theta}\|$ is of the order of $\prod \|A_j\|$ for most values of θ.

The formula allows us to conclude that the mentioned bound of [He] is sharp and also to re-obtain one theorem of [DS].

A similar formula, involving spectral radius, also holds. This motivated us to investigate whether, for cocycles in general, the spectral radius grows like the norm. This problem was posed by Cohen in [Co]. The answer, at least in dimension 2, is no, in general.

2. The formula

Notation: Given a real or complex matrix A, we denote:

$$\|A\| = \sup_{v\neq 0} \frac{\|Av\|}{\|v\|} \quad \text{where } \|\cdot\| \text{ is the euclidean norm.}$$

Also, we denote by $\rho(A)$ the spectral radius, that is, the maximum absolute value of the eigenvalues of A. We have $\rho(A) \leq \|A\|$. We will indicate by $\mathrm{SL}(2, \mathbb{R})$ the group of real two-by-two matrices with unit determinant and $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$. We define:

$$N(A) = \log\left(\frac{\|A\| + \|A\|^{-1}}{2}\right) \quad \text{for } A \in \mathrm{SL}(2, \mathbb{R}).$$

We define some special matrices in $\mathrm{SL}(2, \mathbb{R})$:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{for } \theta \in \mathbb{R}.$$

$$H_c = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{for } c \geq 1.$$

Finally, we indicate by $\mathbb{D}$ the open unit disk in $\mathbb{C}$ and by $\mathbb{S}^1$ its boundary.

Our main formula is:

THEOREM 1: *Let $A_1, \ldots, A_n \in \mathrm{SL}(2, \mathbb{R})$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} N(A_n R_\theta \cdots A_1 R_\theta) d\theta = \sum_{j=1}^{n} N(A_j).$$

Actually, Theorem 1 is a corollary of the formula below:

THEOREM 2: *Let $A_1, \ldots, A_n \in \mathrm{SL}(2, \mathbb{R})$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log \rho(A_n R_\theta \cdots A_1 R_\theta) d\theta = \sum_{j=1}^{n} N(A_j).$$

Theorems 1 and 2 are proved in sections 3 and 4 below.

Notice that $\log\|A\| - \log 2 < N(A) \leq \log\|A\|$. Let's give an interpretation of the quantity $N(A)$ through the following proposition:

PROPOSITION 3: *Let $A \in \mathrm{SL}(2, \mathbb{R})$. Then*

$$N(A) = \frac{1}{2\pi} \int_0^{2\pi} \log\|A(\cos\theta, \sin\theta)\| d\theta.$$

Therefore the number $N(A)$ can be viewed as the "average rate of expansion" of the matrix $A \in \mathrm{SL}(2, \mathbb{R})$.

Proof: By the polar decomposition theorem, one can find numbers $\alpha, \beta \in [0, 2\pi]$ and $c \geq 1$ such that $A = R_\beta H_c R_\alpha$. Moreover, $\|A\| = c$. So we may suppose that $A = H_c$, and we have to prove

$$\frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{c^2 \cos^2\theta + c^{-2} \sin^2\theta} \, d\theta = \log\left(\frac{c + c^{-1}}{2}\right).$$

First we calculate

$$F(b) = \int_0^\pi \log(b^2 \cos^2 \theta + \sin^2 \theta) d\theta.$$

We have

$$F'(b) = 2b \int_0^\pi \frac{d\theta}{b^2 + \tan^2 \theta} = 2b \int_{-\infty}^{+\infty} \frac{dx}{(b^2 + x^2)(1 + x^2)} = \frac{2\pi}{b+1}.$$

(The last integral can be calculated by residues.) The solution of this differential equation with initial condition $F(1) = 0$ is $F(b) = 2\pi \log \frac{1}{2}(b + 1)$. Therefore

$$\int_0^{2\pi} \log \sqrt{c^2 \cos^2 \theta + c^{-2} \sin^2 \theta} d\theta = -2\pi \log c + F(c^2) = 2\pi \log \frac{1}{2}(c + c^{-1}). \quad \blacksquare$$

The corollary of Theorem 1 below is based on an idea from [Kn] and justifies the assertion made in the Introduction:

COROLLARY 4: *Let $A_1, \ldots, A_n \in \mathrm{SL}(2, \mathbb{R})$, $a > 0$ and*

$$E = \left\{ \theta \in [0, 2\pi] : \frac{1}{n} \log \|A_n R_\theta \cdots A_1 R_\theta\| > -a + \frac{1}{n} \sum \log \|A_j\| \right\}.$$

Let ν denote the normalized Lebesgue measure in the circle. Then $\nu(E) \geq 1 - (\log 2)/a$.

Proof: Let $f(\theta) = \frac{1}{n} N(A_n R_\theta \cdots A_1 R_\theta)$ and $M = \frac{1}{n} \sum N(A_j)$. Let

$$F = \{\theta : f(\theta) > M - b\},$$

where $b = a - \log 2$. It is easy to see that $F \subset E$. Since $0 \leq f(\theta) \leq M + \log 2$, we have

$$M = \int f d\nu \leq (M - b)(1 - \nu(F)) + (M + \log 2)\nu(F).$$

This gives

$$\nu(E) \geq \nu(F) \geq \frac{b}{b + \log 2} = 1 - \frac{\log 2}{a}. \quad \blacksquare$$

3. Proof of Theorem 2

The proof is based on complexification methods from [He].

By continuity, we only have to prove the theorem for a dense set of matrices A_i. So we can make the following assumption:

$$B_\theta = A_n R_\theta \cdots A_1 R_\theta \neq \pm I \quad \text{for all } \theta.$$

Define the following complex matrices:

$$S_z = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}^*,$$

$$T_z = \begin{pmatrix} \frac{z^2+1}{2} & -\frac{z^2-1}{2i} \\ \frac{z^2-1}{2i} & \frac{z^2+1}{2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}.$$

We have $T_z = z S_z$ and $S_{e^{i\theta}} = R_\theta$.

Given $A_1, \dots, A_n \in \mathrm{SL}(2, \mathbb{R})$, we define

$$C_z = \prod_{j=1}^n A_j T_z = A_n T_z \cdots A_1 T_z \quad \text{for } z \in \mathbb{C}.$$

LEMMA 5: *There are holomorphic functions* $\lambda_1, \lambda_2 \colon \mathbb{D} \to \mathbb{C}$, *which extend continuously to* $\overline{\mathbb{D}}$, *such that* $\{\lambda_1(z), \lambda_2(z)\}$ *are the eigenvalues of* C_z *and* $|\lambda_2(z)| < |\lambda_1(z)|$ *for every* $z \in \mathbb{D}$.

Lemma 5 implies that $\log \rho(C_z)$ is a harmonic function in the disk $\mathbb{D}$ which extends continuously to the boundary. Moreover,

$$z = e^{i\theta} \quad \Rightarrow \quad C_z = z^n \prod_{j=1}^n A_j R_\theta \quad \Rightarrow \quad \rho(C_z) = \rho\Big(\prod_{j=1}^n A_j R_\theta \Big).$$

Therefore

$$\frac{1}{2\pi} \int_0^{2\pi} \log \rho\Big(\prod_{j=1}^n A_j R_\theta \Big) d\theta = \log \rho(C_0).$$

So the proof of Theorem 2 will be complete once we prove Lemma 5 and the lemma below:

LEMMA 6: *The eigenvalues of* C_0 *are zero and* $\prod_{j=1}^n \frac{1}{2}(\|A_j\| + \|A_j\|^{-1})$.

3.1 Proof of Lemma 5: It is enough to show that the eigenvalues of C_z have different norms for all $z \in \mathbb{D}$. First we obtain the following criteria for identity of their norms:

LEMMA 7: *Let $C \in \mathrm{M}(2, \mathbb{C})$ with $\det C \neq 0$ and let λ_1, λ_2 be the eigenvalues of C. Then $|\lambda_1| = |\lambda_2|$ if and only if*

$$(\mathrm{tr}\, C)^2 / 4 \det C \in [0, 1].$$

Proof: Let $t = \lambda_1 / \lambda_2$. We have $u = (\mathrm{tr}\, C)^2 / 4 \det C = \frac{1}{4}(t + t^{-1} + 2)$. Then

$$|\lambda_1| = |\lambda_2| \Leftrightarrow |t| = 1 \Leftrightarrow \frac{t + t^{-1}}{2} \in [-1, 1] \Leftrightarrow u \in [0, 1]. \quad \blacksquare$$

We have $\det T_z = z^2$ and so $\det C_z = z^{2n}$. Therefore C_z has eigenvalues with equal modulus if and only if $(\mathrm{tr}\, C_z)^2 / 4z^{2n} \in [0, 1]$, that is, if and only if $Q(z) = \mathrm{tr}\, C_z / 2z^n \in [-1, 1]$. So to prove Lemma 5 we must prove that if $z \in \mathbb{D}$ then $Q(z) \notin [-1, 1]$. The idea is that since $Q(z)$ is a rational map of degree at most $2n$, this can be checked by showing that the unit circle 'exhausts' all preimages of $[-1, 1]$. This we will do with a topological argument.

Let $S = Q^{-1}([-1, 1])$.

LEMMA 8: *If $S \cap \mathbb{S}^1$ has at least $2n$ connected components then S is the union of $2n$ sub-intervals of $\mathbb{S}^1$.*

Proof: Notice that $\mathrm{tr}\, C_z$ is a polynomial of degree at most $2n$, so $Q(z)$ and $Q'(z)$ are rational maps of degree at most $2n$. Since $Q(\mathbb{S}^1) \subset \mathbb{R}$ we know that there is at least one 0 of $Q'(z)$ in each component of $\mathbb{S}^1 \smallsetminus S$. In particular there are no zeros of $Q'(z)$ in S, which implies that each connected component of $S \cap \mathbb{S}^1$ is mapped diffeomorphically onto $[-1, 1]$. $\quad \blacksquare$

Define the following sets:

$$X = \mathrm{PSL}(2, \mathbb{R}) \smallsetminus \{I\},$$
$$Y = \{\, A \in X \colon |\,\mathrm{tr}\, A| \leq 2 \,\}. \quad (|\,\mathrm{tr}\,| \text{ is well-defined in } \mathrm{PSL}(2, \mathbb{R}).)$$

LEMMA 9: *There exists a continuous function $F \colon X \to \mathbb{S}^1$ such that $F^{-1}(\{1\}) = Y$ and that the induced homeomorphism $F_{\#} \colon \pi_1(X) \to \pi_1(\mathbb{S}^1)$ is an isomorphism.*

Proof: Let $A \in \mathrm{PSL}(2, \mathbb{R})$. If $A \in Y$ then we define $F(A) = 1$. Otherwise A has two eigendirections $\pm v$ and $\pm w$, where $v, w \in \mathbb{S}^1 \subset \mathbb{C}$, with associated eigenvalues λ and λ^{-1}, where $|\lambda| > 1$. We then define $F(A)$ as v^2 / w^2. It is easy to see that F is continuous at every $A \neq I$.

We have $\pi_1(X) = \mathbb{Z}$ (it is equal to $\pi_1(\mathrm{PSL}(2, \mathbb{R}))$, since $\mathrm{PSL}(2, \mathbb{R})$ is a three-dimensional manifold) and so it is enough to exhibit a closed path γ generating $\pi_1(X)$ such that $F \circ \gamma$ has degree one.

Let $\gamma\colon \mathbb{S}^1 \to X$ be defined as $e^{i\theta} \mapsto R_{\theta/2}M$, where $M \in X$ is symmetric. Using the identities $F(A^T) = \overline{F(A)}$ and $F(R_\theta^{-1} A R_\theta) = F(A)$, we easily see that $F \circ \gamma$ commutes with conjugacy $(F \circ \gamma(\bar{z}) = \overline{F \circ \gamma(z)})$. Furthermore, $e^{i\theta} = 1$ is the only value such that $\gamma(e^{i\theta})$ is symmetric, which is equivalent to $F(\gamma(e^{i\theta})) = -1$. This implies that $F \circ \gamma$ has degree one. ∎

Let $B_\theta = \prod_{j=1}^n A_j R_\theta$. We have $Q(e^{i\theta}) = \frac{1}{2} \operatorname{tr} B_\theta$. Notice that $e^{i\theta} \in S$ if and only if B_θ is an elliptic or parabolic matrix in $\mathrm{SL}(2,\mathbb{R})$.

LEMMA 10: $S \cap \mathbb{S}^1$ has at least $2n$ connected components.

Proof: Let $g\colon \mathbb{S}^1 \to \mathrm{PSL}(2,\mathbb{R})$ be defined by $e^{i\theta} \mapsto R_{\theta/2}$. This is clearly a generator of the fundamental group of $\mathrm{PSL}(2,\mathbb{R})$.

Notice that $e^{i\theta} \mapsto B_\theta$ can also be seen as a path in $\mathrm{PSL}(2,\mathbb{R})$ and it follows from the definition that it is homotopic to g^{2n} (by the fact that $\mathrm{PSL}(2,\mathbb{R})$ is a group). Now we use the assumption made at the beginning: for all θ, $B_\theta \neq \pm I$.

In this case, the path $F \circ B_\theta$ has degree $2n$ and therefore the preimage of 1 has at least $2n$ connected components. This set coincides with $S \cap \mathbb{S}^1$. ∎

Lemmas 8 and 10 imply that $S \subset \mathbb{S}^1$, and Lemma 5 is proved. ∎

3.2 Proof of Lemma 6: We will list some facts to be used:
(1) One can find numbers $\alpha_j, \beta_j \in [0, 2\pi]$, $c_j \geq 1$ such that $A_j = R_{\beta_j} H_{c_j} R_{\alpha_j}$ for each j. Moreover, $\|A_j\| = c_j$.
(2) $A, B \in \mathrm{SL}(2,\mathbb{R}) \quad \Rightarrow \quad \rho(AB) = \rho(BA)$.
(3) For every $\theta \in \mathbb{R}$, $R_\theta T_0 = T_0 R_\theta = e^{-i\theta}T_0$.

Part (1) is the polar decomposition theorem. For (2), notice that the spectral radius depends only on the trace. For (3), we use that $S_z S_w = S_{zw}$. This implies $T_z T_w = T_{zw}$ and

$$R_\theta T_0 = S_{e^{i\theta}} T_0 = e^{-i\theta} T_{e^{i\theta}} T_0 = e^{-i\theta} T_0.$$

Using (1), (2) and (3), we obtain

$$\rho(C_0) = \rho\left(\prod_{j=1}^n A_j T_0 \right) = \rho\left(\prod_{j=1}^n T_0 H_{c_j} \right).$$

Each matrix $T_0 H_{c_j}$ has an eigenvector $(-i, 1)$ with corresponding eigenvalue $\frac{1}{2}(c_j + c_j^{-1})$. Therefore $\prod_{j=1}^n T_0 H_{c_j}$ has an eigenvalue

$$\prod_j (c_j + c_j^{-1})/2,$$

while C_0 is not invertible. This proves Lemma 6 and hence Theorem 2.   ∎

4. Proof of Theorem 1

Let $B_\theta = A_n R_\theta \cdots A_1 R_\theta$. Then, fixing θ we have, by Theorem 2,

$$\frac{1}{2\pi} \int_0^{2\pi} \log \rho(B_\theta R_{\theta'}) d\theta' = N(B_\theta).$$

On the other hand, fixing θ' we have, again by Theorem 2,

$$\frac{1}{2\pi} \int_0^{2\pi} \log \rho(B_\theta R_{\theta'}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log \rho(A_n R_\theta \cdots (A_1 R_{\theta'}) R_\theta) d\theta$$

$$= N(A_1 R_{\theta'}) + \sum_{j=2}^n N(A_j) = \sum_{j=1}^n N(A_j).$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} N(B_\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \log \rho(B_\theta R_{\theta'}) d\theta d\theta' = \sum_{j=1}^n N(A_j).$$

This proves Theorem 1.   ∎

Remark: Inversely, Theorem 2 could be quickly deduced from Theorem 1, using

$$\log \rho(A) = \lim_{n \to \infty} \frac{1}{n} \log \|A^n\| = \lim_{n \to \infty} \frac{N(A^n)}{n}.$$

5. Herman's inequality re-obtained

Let (X, μ) be a probability space and $T: X \to X$ an ergodic transformation. Let $A: X \to \mathrm{SL}(2, \mathbb{R})$ be a measurable function satisfying the integrability condition

$$\int \log \|A\| d\mu < \infty.$$

We denote for $x \in X$ and $n \in \mathbb{N}$,

$$A^n(x) = A(T^{n-1}x) \cdots A(x).$$

The function A is called a **linear cocycle**. In these conditions, there exists (see [FK] or [Le]) a number $\lambda^+(A) \geq 0$, called the **upper Lyapunov exponent**, such that

$$\lambda^+(A) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\| \quad \text{for } \mu\text{-a.e. } x \in X.$$

For $\theta \in \mathbb{R}$, we define a cocycle AR_θ by $(AR_\theta)(x) = A(x)R_\theta$. Clearly, $\theta \mapsto \lambda^+(AR_\theta)$ is a measurable function.

We now state Herman's inequality:

THEOREM 11 ([He], §6.2, see also [Kn]): *If T, μ and A are as above then*

$$\frac{1}{2\pi} \int_0^{2\pi} \lambda^+(AR_\theta)d\theta \geq \int_X \log\Big(\frac{\|A(x)\| + \|A(x)\|^{-1}}{2} \Big) d\mu(x).$$

Remark: Herman's inequality was stated in a different (but equivalent) way, involving the Iwasawa decomposition.

We will re-obtain Theorem 11 and also show that equality holds.

THEOREM 12: *If T, μ and A are as above then*

$$\frac{1}{2\pi} \int_0^{2\pi} \lambda^+(AR_\theta)d\theta = \int_X \log\Big(\frac{\|A(x)\| + \|A(x)\|^{-1}}{2} \Big) d\mu(x).$$

Proof: Recall that $N(A) \leq \log\|A\| < \log 2 + N(A)$. By Theorem 1,

$$\sum_{j=0}^{n-1} N(A(T^j(x))) \leq \frac{1}{2\pi} \int_0^{2\pi} \log\|(AR_\theta)^n(x)\|d\theta \leq \log 2 + \sum_{j=0}^{n-1} N(A(T^j(x))).$$

Therefore, by Birkhoff's theorem,

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{n} \log\|(AR_\theta)^n(x)\|d\theta = \int N(A(x))d\mu(x) \quad \text{for a.e. } x.$$

To finish the proof we must check that Dominated Convergence applies. We have

$$0 \leq \frac{1}{n} \log\|(AR_\theta)^n(x)\| \leq \frac{1}{n} \sum_{j=0}^{n-1} \log\|A(T^j x)\| = f_n(x).$$

$\{f_n\}$ is the sequence of Birkhoff means of the function $\log\|A\| \in L^1(\mu)$. In particular, $\{f_n(x)\}$ is bounded for a.e. x. ∎

Example 13: Consider the cocycle ([He], § 4.1) where $T: \mathbb{S}^1 \to \mathbb{S}^1$ is an (uniquely ergodic) irrational rotation, $A: \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{R})$ is given by $A(e^{it}) = H_c R_t$ and $c \geq 1$ is fixed. We have $(AR_\theta)^n(z) = A^n(e^{i\theta}z)$ and therefore $\lambda^+(A) = \lambda^+(AR_\theta)$ for all θ. It follows from Theorem 12 that $\lambda^+(A) = \log\Big(\frac{1}{2}(c + c^{-1}) \Big)$.

6. A theorem by Dedieu and Shub re-obtained

We will use Proposition 3 and Theorem 2 (in the case $n = 1$) to give another proof of the following theorem by Dedieu and Shub:

THEOREM 14 ([DS]): *Let μ be a probability measure in $\mathrm{SL}(2, \mathbb{R})$ such that the integral $\int \log \|A\| d\mu(A)$ is finite. Suppose that μ is invariant by rotations, that is, $R_\theta^* \mu = \mu$ for all θ. Let $A_1, A_2, \ldots \in \mathrm{SL}(2, \mathbb{R})$ be independent random matrices with law μ and consider the associated upper Lyapunov exponent:*

$$\lambda^+ = \lim_{n \to +\infty} \frac{1}{n} \log \|A_n \cdots A_1\| \quad (w.p.\ 1).$$

Then

$$\lambda^+ = \int_{\mathrm{SL}(2,\mathbb{R})} \log \rho(A) d\mu(A).$$

Proof:

$$\lambda^+ = \int \int_0^{2\pi} \log \|Ae^{i\theta}\| \frac{d\theta}{2\pi} d\mu(A) \qquad \text{(by Furstenberg's formula, see [Le])}$$

$$= \int N(A) d\mu(A) \qquad \text{(by Proposition 3)}$$

$$= \int \int_0^{2\pi} \log \rho(AR_\theta) \frac{d\theta}{2\pi} d\mu(A) \qquad \text{(by Theorem 2 with } n = 1)$$

$$= \int_0^{2\pi} \int \log \rho(AR_\theta) d\mu(A) \frac{d\theta}{2\pi} \qquad \text{(since } \log \rho(AR_\theta) \leq \log \|A\| \in L^1)$$

$$= \int \log \rho(A) d\mu(A) \qquad \text{(since } \mu \text{ is invariant by rotations).} \quad \blacksquare$$

7. Growth of the spectral radius

Let X, μ, T and A be as in section 5. In view of our results, it is somewhat natural to ask about the behavior of the spectral radius of the matrix $A^n(x)$ when $n \to \infty$. This question was already raised in [Co]. We have the following result:

THEOREM 15: *Suppose T is invertible. Then for μ-a.e. $x \in X$,*

$$\limsup_{n \to \infty} \frac{1}{n} \log \rho(A^n(x)) = \lambda^+(A).$$

Before giving the proof, we point out that in general the limit of $\frac{1}{n} \log \rho(A^n(x))$ does not exist. Furthermore, the relation

$$(*) \qquad \limsup_{n \to \infty} \frac{1}{n} \int \log \rho(A^n) d\mu = \lambda^+(A)$$

is in general *false*, as is shown by the following:

Example 16:   Let $X = \{0,1\}^{\mathbb{Z}}$, μ be the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure and let $T \colon X \to X$ be the left shift. We define a cocycle $A \colon X \to \mathrm{SL}(2, \mathbb{R})$ by:

$$A(\{x_i\}_{i \in \mathbb{Z}}) = \begin{cases} H = H_2 & \text{if } x_0 = 1, \\ I & \text{if } (x_{-1}, x_0, x_1) = (0,0,0) \text{ or } (1,0,1), \\ R = R_{\pi/2} & \text{if } (x_{-1}, x_0, x_1) = (1,0,0) \text{ or } (0,0,1). \end{cases}$$

Given any sequence $x = \{x_i\}_{i \in \mathbb{Z}}$, split it in minimal blocks starting with 1, as, for instance,

$$\ldots (10)(1)(1000)(100)(10)(100000)(1) \ldots$$

The corresponding splitting for the sequence $\{A(T^i(x))\}$ is, in this case,

$$\ldots (HI)(H)(HRIR)(HRR)(HI)(HRIIIR)(H) \ldots$$

The product of the matrices in each block is always $\pm H$. It follows that $\lambda^+(A) = \frac{1}{2}\log 2$. On the other hand, making substitutions $R^2 = -I$ in the product $A^n(x)$, we obtain one of the possibilities: $\pm H^k$, $\pm H^k R$, $\pm R H^k$ or $\pm R H^k R$. Since $\rho(H^k R) = \rho(R H^k) = 1$, we have $\rho(A^n(x)) = 1$ infinitely often for a.e. x. Besides, it's not hard to show that $(*)$ does not hold.

Proof of Theorem 15:   We may regard the problem as being posed in $\mathrm{PSL}(2, \mathbb{R})$ instead of $\mathrm{SL}(2, \mathbb{R})$. Suppose that $\lambda^+(A) > 0$ (otherwise there is nothing to prove). Consider (see [Le]) the Oseledets splitting $\mathbb{R}^2 = E^+(x) \oplus E^-(x)$, defined for a.e. $x \in X$, where E^+ (resp. E^-) is associated to the exponent $\lambda^+(A)$ (resp. $-\lambda^+(A)$). By Oseledets' theorem,

$$\lim_{n \to \infty} \frac{1}{n} \log \sin \measuredangle(E^+(T^n x), E^-(T^n x)) = 0.$$

For each x, take $B(x) \in \mathrm{PSL}(2, \mathbb{R})$ that sends the direction $\mathbb{R}(1,0)$ (resp. $\mathbb{R}(0,1)$) to the direction $E^+(x)$ (resp. $E^-(x)$). This defines a.e. a measurable function $B \colon X \to \mathrm{PSL}(2, \mathbb{R})$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|B(T^n x)\| = 0 \quad \text{for a.e. } x.$$

We claim that

$$\liminf_{n \to \infty} \|B(x)^{-1} B(T^n x) - I\| = 0 \quad \text{for a.e. } x.$$

To prove it, let $\varepsilon > 0$. Consider a countable cover of $\mathrm{PSL}(2, \mathbb{R})$ by open sets

$$U_j = \{M \in \mathrm{PSL}(2, \mathbb{R}) \colon \|M - M_j\| < \delta_j\}, \quad \text{where } 2\delta_j(\|M_j\| + \delta_j) < \varepsilon.$$

Define $V_j = B^{-1}(U_j) \subset X$ and

$$\tilde{V}_j = \{\, x \in V_j \colon T^n(x) \in V_j \text{ for infinitely many } n \in \mathbb{N}\}.$$

By Poincaré's recurrence theorem, $\mu(\tilde{V}_j) = \mu(V_j)$. If $x \in \tilde{V}_j$ then, for infinitely many $n \in \mathbb{N}$, we have

$$\|B(x)^{-1}B(T^n x) - I\| \le \|B(T^n x) - B(x)\| \cdot \|B(x)\| < 2\delta_j(\|M_j\| + \delta_j) < \varepsilon.$$

Therefore $\liminf \|B(x)^{-1}B(T^n x) - I\| \le \varepsilon$ for every x in the full measure set $\bigcup \tilde{V}_j$. This proves the claim.

To prove the Theorem it's enough (since $\rho + \rho^{-1} = \max\{|\operatorname{tr}|, 2\}$) to show that

$$\limsup_{n \to \infty} \frac{1}{n} \log |\operatorname{tr} A^n(x)| = \lambda^+(A).$$

By construction, the matrix $H(x) = B(Tx)^{-1}A(x)B(x)$ is diagonal. We have $A^n(x) = B(T^n x)H^n(x)B(x)^{-1}$ and, in particular, $\lim \frac{1}{n} \log \|H^n(x)\| = \lambda^+(A)$. Write $B(x)^{-1}B(T^n x) = (b_{ij}(n, x))_{i,j=1,2}$. For a.e. x, we know that there are infinitely many $n \in \mathbb{N}$ such that

$$|b_{11}(n, x) - 1|, |b_{22}(n, x) - 1| < \frac{1}{2}.$$

The matrices $A^n(x)$ and $B(x)^{-1}B(T^n x)H^n(x)$ have the same trace, so

$$
\begin{aligned}
|\operatorname{tr} A^n(x)| &= |\operatorname{tr}(B(x)^{-1}B(T^n x)H^n(x))| \\
&= \left| [b_{11}(n, x) \cdot \|H^n(x)\| + b_{22}(n, x) \cdot \|H^n(x)\|^{-1}] \right| \\
&> \frac{1}{2}\|H^n(x)\| - \frac{3}{2}\|H^n(x)\|^{-1}.
\end{aligned}
$$

The result follows.   ∎

References

[Bo]   J. Bochi, *Genericity of zero Lyapunov exponents*, Preprint 2000, available at www.preprint.impa.br.

[BPSW]   K. Burns, C. Pugh, M. Shub and A. Wilkinson, *Recent results about stable ergodicity*, in *The Seattle Conference of Smooth Ergodic Theory*, Proceedings on Symposia in Pure Mathematics, to appear.

[Co]   J. E. Cohen, *Open problems — The spectral radius of a product of random matrices*, Contemporary Mathematics **50** (1986), 329–330.

[DS]   J.–P. Dedieu and M. Shub, *On random and mean exponents for uni-
        tarily invariant probability measures on* $\mathbb{GL}_n(\mathbb{C})$, Preprint 2001, available at
        `www.research.ibm.com/people/s/shub`.

[FK]   H. Furstenberg and H. Kesten, *Products of random matrices*, Annals of Math-
        ematical Statistics **31** (1960), 457–469.

[He]   M. R. Herman, *Une méthode pour minorer les exposants de Lyapounov et
        quelques exemples montrant le caractère local d'un théorème d'Arnold et de
        Moser sur le tore de dimension 2*, Commentarii Mathematici Helvetici **58**
        (1983), 453–502.

[Kn]   O. Knill, *Positive Lyapunov exponents for a dense set of bounded measurable
        SL(2, R)-cocycles*, Ergodic Theory and Dynamical Systems **12** (1992), 319–
        331.

[Le]   F. Ledrappier, *Quelques propriétés des exposants caractéristiques*, Lecture
        Notes in Mathematics **1097**, Springer-Verlag, Berlin, 1982, pp. 305–396.